DETERMINATION OF THE UNLOADING WAVE IN A PARTICULAR CASE

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The present paper concerns the determination of the unloading wave in an elastic-plastic rod whose material has the following properties with respect to uniaxial strain.

The loading branch of the compression stress-strain curve is subject to the conditions

$$\frac{d^2\sigma}{d\varepsilon^2} \leqslant 0, \qquad \left(\frac{d\sigma}{d\varepsilon}\right)\bigg|_{\varepsilon=0} < +\infty$$

but is otherwise arbitrary; unloading occurs with constant density. We shall show that in this case the problem of determining the parameters of the unloading wave reduces to the integration of an ordinary first-order differential equation (*). This equation is integrable in quadratures and, in a particular case, in elementary functions. Some examples will be considered.

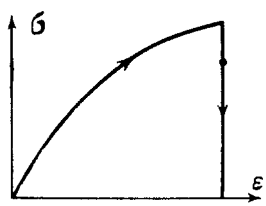

Fig. 1

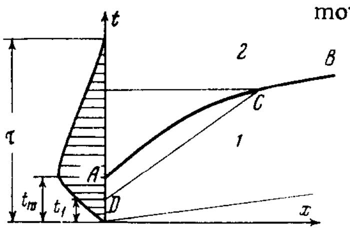

Fig. 2

1. Let us consider a homogeneous semi-infinite straight rod of constant cross section made of an elastic-plastic material. Limiting our attention to plane one-dimensional motions, we shall describe the properties of the material by means of a uniaxial compression stress-strain curve assuming that its loading portion is convex upward (Fig. 1), and that unloading occurs with constant density. The origin will be placed at the free end on the rod axis (x is the Lagrangian coordinate and t is time). A specified load $p(t)$ ($p(t) \geq 0$) acts at the end of the rod ($x = 0$).

Let us assume that $p(t)$ increases monotonously for $0 \leq t \leq t_m$ and decreases monotonously to zero for $t > t_m$. In particular, let us consider the case where $p(t) \equiv 0$ for $t > T > t_m$ (t_m and T are given constants).

With the compression stress-strain diagram under consideration the perturbations propagate in the form of continuous waves (there is no shock wave with the stress and

*) This equation was obtained by G. K. Iagund under the assumption of a piecewise-linear two-part compression stress-strain diagram (unpublished manuscript). A similar problem is considered in [1], although the investigation and conclusions of this paper are only partly duplicated here.

velocity discontinuity). The character of the external force is such that the perturbed region has two parts, i. e. loading region 1 and unloading region 2 (Fig. 2). The boundary between these regions is called the unloading wave (AB, Fig. 2). This boundary is a line in the xt plane. Our problem consists in finding the unloading wave (which is not known in advance) and in determining the state parameters (stress and velocity) on this wave. Having done this, we can readily determine the state of the medium at any point of the xt plane. In the general case finding the unloading wave is no easy problem : this is typical of boundary value problems with an unknown boundary. We shall show, however, that under the above conditions the problem of determining the unloading wave reduces to the integration of an ordinary first-order differential equation .

2 . Let $\sigma(x, t)$ be the normal stress on the transverse cross section, $v(x, t)$ the particle velocity, $\epsilon(x, t)$ the longitudinal strain, and ρ the density. The motion is described by the system of equations

$$\frac{\partial\sigma}{\partial x}+\rho\frac{\partial v}{\partial t}=0, \qquad \frac{\partial v}{\partial x}+\frac{\partial\varepsilon}{\partial t}=0, \qquad \sigma=\sigma(\varepsilon),\ \sigma(0, t)=p(t) \tag{2.1}$$

It is assumed that the compressive stresses and compressive strain are positive.

In region 1 (Fig. 2) the stresses are determined by the loading branch of the diagram. With the stipulated method of loading the motion in this region is described by simple (Reimann) waves. The straight lines

$$x=a(\sigma)(t-t_1), \qquad a^2=\frac{1}{\rho}\frac{d\sigma}{d\varepsilon} \tag{2.2}$$

are characteristics of positive direction: the condition v = const, σ = const is fulfilled along each characteristic. The relation

$$dv=(\rho a(\sigma))^{-1}\,d\sigma \tag{2.3}$$

is valid throughout region 1 (e. g. see [1]) .

In region 2 the strains are constant in each particle. Hence, from (2. 1) we find that

$$\frac{\partial\varepsilon}{\partial t}=0, \qquad \varepsilon=\varepsilon(x), \qquad v=v(t), \qquad \sigma(x, t)=-\rho\frac{\partial v}{\partial t}x+p(t) \tag{2.4}$$

Provided the velocities and stresses on the unloading wave are continuous, $dv_1 = dv_2$ along AB (Fig. 2) . (The subscripts denote the limiting values at both sides of AB).

But

$$dv_1=\frac{\partial v_1}{\partial x}dx+\frac{\partial v_1}{\partial t}dt, \qquad dv_2=\frac{\partial v_2}{\partial t}dt$$

So that

$$\frac{\partial v_2}{\partial t}=\frac{\partial v_1}{\partial t}+\frac{\partial v_1}{\partial x}\frac{dx}{dt}=\frac{\partial v_1}{\partial t}+\frac{\partial v_1}{\partial x}\varphi'(t)$$

Here $x=\varphi(t)$ is the equation of the unloading wave.

Introducing the notation $\sigma^*=\sigma(\varphi(t), t)$, $v^*=v_1(\varphi(t), t)$. , we have

$$\frac{\partial v_2}{\partial t}=\frac{dv^*}{dt} \quad \text{for} \quad \frac{\partial v_2}{\partial t}=\frac{1}{\rho a(\sigma^*)}\frac{d\sigma^*}{dt} \tag{2.5}$$

The continuity of the stresses on the unloading wave and (2. 4) imply that

$$\frac{x_*}{a(\sigma^*)}\frac{d\sigma^*}{dt}+\sigma^*=p(t_*) \tag{2.6}$$

where x_* , t_* are the coordinates of the point C (Fig. 2).

Equation (2. 2) is the equation of the characteristic CD , so that

$$x_*=a(\sigma^*)(t_*-t_1(\sigma^*)) \tag{2.7}$$

Eliminating x from the two latter relations and again replacing t_* by t , we obtain for $\sigma^* = \sigma^*(t)$ the differential equation

$$\frac{d\sigma^*}{dt} = \frac{p(t) - \sigma^*}{t - t_1(\sigma^*)} \qquad \text{for } t_m < t \tag{2.8.1}$$

with the initial condition

$$\sigma^*(t_m) = p(t_m) = \sigma_m$$

Having been found, the function $\sigma^*(t)$ together with Equation (2. 7) determines the equation of the unloading wave. Equation (2. 8. 1) is integrable in quadratures. In fact, it can be rewritten in the form

$$d(t\sigma^*) = p(t)\,dt + t_1(\sigma^*)\,d\sigma^* \tag{2.8}$$

Integration over the limits t_m and t with allowance for the initial condition, yields

$$t\sigma^* = t_m\sigma_m + \int_{t_m}^{t} p(\tau)\,d\tau + \int_{\sigma_m}^{\sigma^*} t_1(\sigma)\,d\sigma \tag{2.9.1}$$

At the same time,

$$\int_{\sigma_m}^{\sigma^*} t_1(\sigma)\,d\sigma = t_1(\sigma^*)\,\sigma^* - t_m\sigma_m - \int_{t_1(\sigma^*)}^{t_m} p_1(t)\,dt$$

where $p_1(t)$ is the given stress at the end of the rod during the loading phase. Thus,

$$[t - t_1(\sigma^*)]\,\sigma^* = \int_{t_1(\sigma^*)}^{t_m} p_1(t)\,dt + \int_{t_m}^{t} p(t)\,dt \tag{2.9.2}$$

Let us denote

$$I(t) = \int_{0}^{t_m} p_1(t)\,dt + \int_{t_m}^{t} p(t)\,dt \qquad (t > t_m)$$

Then

$$[t - t_1(\sigma^*)]\,\sigma^* = I(t) - \int_{0}^{t_1(\sigma^*)} p_1(\tau)\,d\tau \tag{2.9.3}$$

Since $0 < t_1(\sigma^*) < t_m$, $0 \leqslant p_1(t) \leqslant \sigma_m$ and $p_1(t)$ increases monotonously, it follows that

$$\int_{t_1(\sigma^*)}^{t_m} p_1(\tau)\,d\tau > [t_m - t_1(\sigma^*)]\,\sigma^*$$

and from (2. 9. 2) and (2. 9. 3) we have the estimates

$$\frac{1}{t - t_m}\int_{t_m}^{t} p(\tau)\,d\tau < \sigma^*(t) < \frac{I(t)}{t - t_m} \tag{2.10}$$

The first of these inequalities implies that

$$p(t) - \sigma^* < p(t) - \frac{1}{t - t_m}\int_{t_m}^{t} p(\tau)\,d\tau$$

Since $p(t)$ decreases monotonously, then $p(t) - \sigma^* < 0$, and from (2. 8. 1) we conclude that $\sigma^*(t)$ decreases monotonously and tends to zero is $I(\infty) < +\infty$. The asymptotic behavior of $\sigma^*(t)$ as $t \to \infty$ follows from (2. 9. 3),

$$\sigma^*(t) = \frac{I(t)}{t} + \frac{t_1(\sigma^*)\,\sigma^*}{t} - \frac{1}{t}\int_{0}^{t_1(\sigma^*)} p_1(\tau)\,d\tau = \frac{I(t)}{t} + \frac{1}{t}\int_{0}^{\sigma^*} t_1(\sigma)\,d\sigma$$

Specifically,

$$\sigma^*(t) = \frac{I(t)}{t} + o\left(\frac{1}{t}\right) = \frac{I(\infty)}{t} + o\left(\frac{1}{t}\right) \tag{2.11.1}$$

On the other hand, if the loading is instantaneous ($t_m = 0$), the exact equation

$$\sigma^* (t) = t^{-1} I (t) \tag{2.11.2}$$

is valid. From (2.11.2) we see that the maximum stress at the point under consideration is equal (regardless of the loading branch of the compression stress-strain diagram) to the average external stress over the time from the application of the load to the instant of arrival of the stress maximum at the observation point. The time required for the attainment of the stress maximum is usually easy to determine experimentally. Hence, the above fact can be used in treating experimental results. In the general case when $t_m > 0$, the stress $\sigma^*(t)$ on the unloading wave is equal to the average applied stress over the time interval $(t_1(\sigma^*), t)$ (see (2.9.2)). It is important to note that the stress on the unloading wave considered as a function of time is independent of the compression stress-strain diagram, since the equation of the unloading wave depends on the latter.

The above conclusions can be used to find the asymptote to the unloading wave and the tangent at its initial point.

Let us suppose that the equation of the asymptote is of the form $X = k(t - t_0)$, where k and t_0 are to be determined. From (2.7) we find that

$$x - X = [a (\sigma^*) - k] t + kt_0 - a (\sigma^*) t_1 (\sigma^*)$$

Using the expansion

$$a (\sigma^*) = a (0) + a' (0) \sigma^* + \ldots$$

and applying the usual rules, we find that

$$k = a (0), \qquad t_0 = - \frac{a' (0)}{a (0)} I (\infty) \tag{2.12}$$

The equation of the tangent to the unloading wave at the point $(0, t_m)$ will be assumed to be of the form

$$x = k_0 (t - t_m)$$

where k_0 is to be determined.

Let the expansions (2.13)

$$p (t) = \sigma_m - p_1 (t - t_m) + \ldots, \ t_1 (\sigma^*) = t_m + q_1 (\sigma^* - \sigma_m) + \ldots, \sigma^* = \sigma_m - c_1 (t - t_m) + \ldots$$

be valid in the neighborhood of the point $(0, t_m)$.

Here p_1, q_1 are known and $p_1 > 0$, $q_1 > 0$; c_1 is a coefficient to be determined. Substituting (2.13) into (2.8) and taking the limit as $t \to t_m$, we obtain

$$- c_1 = \frac{c_1 - p_1}{1 + q_1 c_1} \tag{2.14}$$

By the condition of the problem $t = t_1(\sigma^*) \geq 0$, so that $1 + q_1 c_1 \geq 0$.

If we assume that $c_1 < 0$, Equation (2.14) is contradictory. Hence, $c_1 \geq 0$, and we obtain the value

$$c_1 = \frac{p_1}{1 + \sqrt{1 + p_1 q_1}}$$

for this coefficient.

On the basis of (2.7) we obtain the equation of the required tangent

$$x = a (\sigma_m) \sqrt{1 + p_1 q_1} (t - t_m) \tag{2.15}$$

Let us consider briefly the particular case where the applied stress increases linearly.

$$t_1 (\sigma) = t_m (\sigma / \sigma_m)$$

For σ^* we obtain

$$\sigma^* (t) = \frac{2I (t)}{t + \sqrt{t^2 - 2qI (t)}}, \qquad q = \frac{t_m}{\sigma_m} \tag{2.16}$$

3. Let us consider an example which illustrates the effect of the compression stress-strain diagram on the decline of the maximum stress with distance.

Let the compression stress-strain diagram be given by Equation

$$\frac{\varepsilon}{\varepsilon_m} = \frac{1}{n}\frac{\sigma}{\sigma_m} + \frac{n-1}{n}\left(\frac{\sigma}{\sigma_m}\right)^{n+1} \tag{3.1}$$

for σ and ϵ lying in the ranges

$$0 \leqslant \sigma \leqslant \sigma_m, \quad 0 \leqslant \varepsilon \leqslant \varepsilon_m$$

The corresponding curves are shown in Fig. 3.

Now let us consider the case where the applied load is of the form

$$p(t) = \begin{cases} \sigma_m(1 - t/\tau), & 0 < t \leqslant \tau \\ 0 & t < 0,\ t > \tau \end{cases}$$

We then use (2.11.2) to obtain the following expression for the stress $\sigma^*(t)$ on the unloading wave:

$$\frac{\sigma^*(t)}{\sigma_m} = \begin{cases} 1 - t/2\tau, & 0 \leqslant t \leqslant \tau \\ \tau/2t, & t \geqslant \tau \end{cases} \tag{3.2}$$

The equation of the unloading wave on the xt plane is then

$$x^* = \tau\sqrt{\frac{\sigma_m}{\rho\varepsilon_m}}\sqrt{\frac{n}{1 + (n^2 - 1)(\sigma^*/\sigma_m)^n}}\,\frac{t}{\tau} \tag{3.3}$$

Eliminating t from Equations (3.2) and (3.3), we obtain the relationship between

$$\sigma^\circ = \frac{\sigma^*(t)}{\sigma_m}, \qquad x^\circ = \frac{x^*}{\tau}\sqrt{\frac{\rho\varepsilon_m}{\sigma_m}}$$

Values of σ° for several values of t, as well as values of x° computed for the same values of t and various n (Fig. 4) are given below

$t =$	0.250	0.500	0.750	1.000	1.500	2.000	3.000	5.000	
$\sigma^\circ =$	0.875	0.750	0.625	0.500	0.333	0.250	0.167	0.100	
$x^\circ =$	0.250	0.500	0.750	1.000	1.500	2.000	3.000	5.000	$(n = 1)$
$x^\circ =$	0.195	0.431	0.720	1.069	1.838	2.592	4.074	6.970	$(n = 2)$
$x^\circ =$	0.160	0.417	0.827	1.436	2.756	3.888	5.964	9.990	$(n = 4)$
$x^\circ =$	0.149	0.523	1.351	2.534	4.222	5.654	8.484	14.140	$(n = 8)$
$x^\circ =$	0.209	1.486	3.300	4.471	6.708	8.944	13.416	22.360	$(n = 20)$

The computed results show that changes in the compression stress-strain diagram (changes in the parameter n) have a weak effect on the law of decay of the maximum stress at small distances only; for $x^\circ > \frac{1}{2}$ the effect of the diagram becomes noticeable and remains

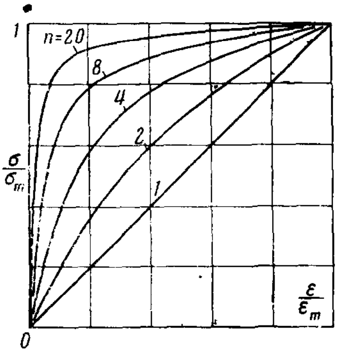

Fig. 3

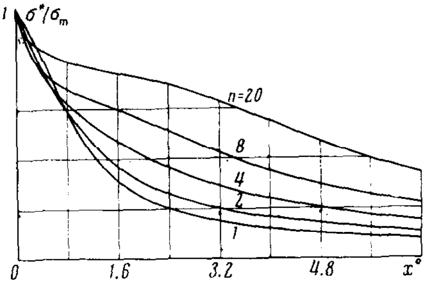

Fig. 4

evident all the way to the asymptotic form for large times and distances, for which we have

$$\sigma^\circ = \frac{\sqrt{n}}{2x^\circ}$$

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A RECIPROCAL THEOREM FOR DYNAMIC PROBLEMS OF THE THEORY OF ELASTICITY

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A reciprocal principle for dynamic problems of the theory of elasticity is given in the papers [1 to 3]. In this paper, a more general reciprocal principle is presented for the case in which displacement as well as traction boundary conditions are imposed on an elastic body. In contrast to the generally used method of derivation of the reciprocal theorem in dynamics, employing the Laplace transform and Betti's law, the theorem of reciprocity is derived here from a variational principle.

We note that in [4] the opposite route is used for static problems, i. e. the variational principles of the theory of elasticity are deduced from the reciprocal theorem.

Let u_k, v_k, π_k be the components of the displacement, velocity, and generalized momentum vectors: ε_{ik} and σ^{ik} the components of the strain and stress tensors: E^{iklm} the components of the tensor of elastic constants: X^k and P^k the components of the body force and external traction vectors; U^k the components of the specified displacements; and ρ, V and S the density, volume and surface of the elastic body.

The solution of a dynamic problem of the theory of elasticity reduces to the integration of the equations of motion

$$\nabla_i\sigma^{ik} + X^k = \frac{\partial \pi^k}{\partial t} \qquad (1)$$

where

$$\sigma^{ik} = E^{ikjl}\varepsilon_{jl}, \qquad \pi_k = \rho v_k, \qquad \varepsilon_{ik} = \frac{1}{2}(\nabla_i u_k + \nabla_k u_i), \qquad v_k = \frac{\partial u_k}{\partial t} \qquad (2)$$